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On the magnetic moment of an electron gas in an inhomogeneous magnetic field

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Abstract. The magnetic behaviour of a Dirac electron gas in the presence of the inhomogeneous magnetic field, $H\ \mathrm{sech}^2(ay)$, is studied. Using the single-particle energy eigenvalue of a Dirac electron, the magnetic interaction energy density is written down, from which an explicit expression for the magnetic moment is derived. The magnetic moment density is evaluated numerically in the degeneracy limit for several values of the magnetic field strength and the chemical potential. The transition from para- to diamagnetism exhibited by the electron gas in the homogeneous magnetic field is found to persist in the inhomogeneous field also. Further, the distinct possibility of spontaneous magnetisation (i.e. ferromagnetic behaviour) of the electron gas in this inhomogeneous field is discussed.

1. Introduction

The knowledge of the behaviour of an electron gas in a static homogeneous magnetic field (HMF) of strength of the order of 10^{13}G is of great importance in the study of certain gravitationally collapsed bodies like the neutron stars. Recently, the thermodynamic and magnetic behaviour of the electron gas in such large fields has been studied by Canuto and Chiu (1968a, b, c). These authors calculated the macroscopic magnetic moment of the gas which determines the magnetic property of the system. They found that the magnetic moment is positive for the first few levels and then switches over to a negative value, the switch over to diamagnetism being dependent on the field strength as well as the chemical potential. However, no spontaneous magnetisation takes place. The maximum magnetic moment is only about 10^{-3} times the strength of the inducing magnetic field. Thus the non-interacting electron gas in a HMF does not display ferromagnetic behaviour. Also the inclusion of the anomalous magnetic moment does not alter the above conclusions (Chiu *et al* 1968).

It is an indisputable fact that most of the fields in nature, both terrestrial and celestial, are inhomogeneous in character (Pikel'ner and Khokhlova 1973). The motion of charged particles in inhomogeneous magnetic fields (IMFs) is an increasingly important topic, e.g. in geophysics, solar physics and thermonuclear research (Jackson 1975). In a previous paper (Achuthan *et al* 1979) we considered the striking possibility of electron–positron pair creation in the IMFs. In view of this and other considerations, it is quite reasonable to expect that the study of the thermodynamic, quantum electrodynamic and weak interaction properties of the Dirac electron in IMFs could yield further interesting and possibly startling results.

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In the present paper, we study the magnetic properties of the Fermi gas in the IMF, $H\ \mathrm{sech}^2(ay)$ where a is the inhomogeneity parameter. In § 2, the expression for the magnetic moment of the degenerate electron gas is given. In § 3 we present the numerical results of our calculations which show the paramagnetic to diamagnetic transition as well as a possibility of spontaneous magnetisation of the electron gas.

2. Magnetic moment of an electron gas

2.1. Energy eigenvalues

The inhomogeneous magnetic field

$$H_x = H_y = 0, \qquad H_z(y) = H\ \mathrm{sech}^2(ay) \tag{1}$$

is derived from the vector potential

$$A_x(y) = -(H/a)\tanh(ay), \qquad A_y = A_z = 0. \tag{2}$$

In the above, H is the field strength at $y=0$. With the vector potential defined in (2), the two-component Dirac equation for an electron can be solved exactly (Stanciu 1967). The eigenfunctions are obtained in terms of the Jacobi polynomials and the energy eigenvalues are given by

$$E_N = mc^2\{1+(p_z/mc)^2+(p_x/mc)^2+2N(H/H_c) - (a\lambda\!\!\!{}^-_cN)^2-(p_x/mc)^2[1-(a\lambda\!\!\!{}^-_c)^2N/(H/H_c)]^{-2}\}^{1/2}, \tag{3}$$

where p_x and p_z are the electron momenta along the x and z directions, respectively. The quantum number N is defined by

$$N = n + \tfrac{1}{2}s + \tfrac{1}{2}, \tag{4}$$

where $n = 0, 1, 2, \ldots$ and $s = +1\ (-1)$ for the state with spin up (down). Further,

$$H_c = m^2c^3/e\hbar = 4.414\times 10^{13}\,\mathrm{G} \tag{5}$$

is the Landau critical magnetic field beyond which quantum electrodynamics is supposed to break down (Landau and Lifshitz 1975) and $\lambda\!\!\!{}^-_c = \hbar/mc$ is the electron Compton wavelength divided by 2π. In order that bound state solutions exist for the Dirac equation in the field (1), the quantum number N must satisfy the inequality

$$N \leqslant (H/H_c)/(a\lambda\!\!\!{}^-_c)^2 - |(p_x/mc)/(a\lambda\!\!\!{}^-_c)^3|^{1/2}. \tag{6}$$

Note that when the inhomogeneity parameter a is set to zero the magnetic field (1) becomes homogeneous, the restriction on the quantum number N is removed and also the energy eigenvalue (3) reduces to

$$E_N = mc^2[1+(p_z/mc)^2+2N(H/H_c)]^{1/2} \tag{7}$$

which is the energy of a Dirac electron in a homogeneous magnetic field (Johnson and Lippmann 1949).

We would like to make a few observations before proceeding further.

(i) The appearance of the product term involving the quantum number N and the continuous variable p_x in (3) should not be surprising. A similar feature is seen in the case of the motion of an electron in the presence of crossed homogeneous magnetic and electric fields (Canuto and Chiuderi 1969). The explicit appearance of p_x is due to the drift motion of the electron (in the classical picture) along the direction perpendicular to both the field strength and the gradient of the field (Jackson 1975).

(ii) Although the inhomogeneity parameter a can be arbitrary, physical considerations impose certain limitations on the range of values of a. We know that $\operatorname{sech}^2(ay)$ is a rapidly decreasing function of ay. For instance, when $ay = 0$, $\operatorname{sech}^2(ay) = 1$ and when $ay = 20$, $\operatorname{sech}^2(ay) \sim 10^{-17}$. Therefore, for H of the order of H_c, $H_z(y)$ becomes of the order of 10^{-4}, when $ay = 20$. If we choose large values of a, the field strength becomes vanishingly small even for small values of y. We choose two particular values of a, one for the neutron star dimensions (~ 10 km) and another for the laboratory dimensions (~ 1 m) such that the field strength, $H_z(y)$ is appreciable over the whole region under consideration. The field does not vanish, even though it decreases with ay, and hence the question of matching at the boundaries does not arise.

(iii) The above constraint on the parameter a and also its appearance along with λ_c in the energy expression (3) make the effect of the inhomogeneity in the energy to be practically zero. However, we shall see later that the space dependence of the magnetic moment density that arises due to the inhomogeneous nature of the field makes a considerable difference from the case of homogeneous magnetic fields. In another IMF under study, namely

$$H_z(r) = b/r^a,$$

this problem of the insensitivity of the energy and hence the thermodynamic quantities due to changes in the inhomogeneity does not arise. The results of our investigation in this IMF will be presented elsewhere.

2.2. *Definition of the magnetic moment*

The magnetic moment $\boldsymbol{M}$ of an electron gas of volume V in the presence of a HMF is defined through the relation

$$\Omega = -\boldsymbol{M} \cdot \boldsymbol{H}, \tag{8}$$

where Ω is the thermodynamic potential. In the particular case when the magnetic field is along the z direction we can write

$$M_z = -\partial\Omega/\partial H_z. \tag{9}$$

The above definition for the magnetic moment cannot as such be carried over to the inhomogeneous case. For an arbitrary IMF along the z direction, $H_z(x, y, z)$, the thermodynamic potential can be written as

$$\Omega = -\int_V \sigma(x, y, z)\, \mathrm{d}V, \tag{10}$$

where

$$\sigma(x, y, z) = -I_z(x, y, z) H_z(x, y, z) \tag{11}$$

is the thermodynamic potential density. In (11), $I_z(x, y, z)$ is the magnetic moment density and it is given by

$$I_z(x, y, z) = -\partial\sigma(x, y, z)/\partial H_z(x, y, z) \tag{12}$$

in analogy with (9).

The total magnetic moment $M = M_z$ of the gas is obviously

$$M = \int_V I_z(x, y, z)\,\mathrm{d}V. \tag{13}$$

In the case of the HMF, where I_z is independent of the coordinates, (10) simplifies to

$$\Omega = -H \int_V I_z\,\mathrm{d}V = -HI_z V \tag{14}$$

with the total magnetic moment M, identified by

$$M = I_z V, \tag{15}$$

as is to be expected.

2.3. *Thermodynamic potential*

The thermodynamic potential Ω of a Fermi gas is given by (Landau and Lifshitz 1959)

$$\Omega = -kT \sum_j \log[1 + \exp\beta(\mu - \varepsilon_j)]. \tag{16}$$

Here $\beta = mc^2/kT$, with k the Boltzmann constant and T the absolute temperature. Further, μ is the chemical potential plus electron rest energy and ε_j is the energy of the electron in the jth quantum state, both expressed in units of mc^2. The summation in (16) indicates summation over the discrete quantum numbers n and s and integration over the momentum variables.

The contribution to the summation from the continuous variable p_z is

$$\frac{V^{1/3}}{2\pi\hbar} \int_{-\infty}^{+\infty} \mathrm{d}p_z.$$

The summand in (16) includes the density of state factor ω_N which brings in its wake integrals over p_x and p_y during its computation. In the case of the HMF these integrals can be performed straightaway. But because of the presence of p_x in the integrand the p_x integral cannot be performed and we have to have recourse to an operator definition of ω_N as given below in equation (18). The definition is justified by the recovery of the corresponding expression for the homogeneous case.

We compute ω_N in the usual way (Huang 1963). In the absence of the magnetic field the momenta p_x and p_y form a continuous distribution. When the field is turned on, the energies of the states are changed by finite amounts and a certain number of states coalesce to form the Landau levels. In particular, for the IMF under consideration, by comparing the energy spectrum of a free electron with equation (3) we find that in the two-dimensional space spanned by p_x and p_y the energy surfaces are ellipses given by

$$\eta^2/[b(N)]^2 + \zeta^2/[g(N)]^2 = 1, \tag{17}$$

where

$$\eta = p_x/mc, \qquad \zeta = p_y/mc,$$

$$b(N) = [1 - (a\lambda_c)^2 N(H/H_c)]g(N),$$

$$g(N) = [2NH/H_c - (a\lambda_c N)^2]^{1/2}.$$

In the case of the HMF equation (17) reduces to the equation of a circle: $\eta^2+\zeta^2=2NH/H_c$.

The factor ω_N is obtained by considering the 'area' of the phase space (x, y, η, ζ) corresponding to the moments η and ζ lying between two ellipses N and $N+1$ and dividing by the unit of area $(2\pi\lambda_c)^2$:

$$\omega_N=\frac{1}{4(\pi\lambda_c)^2}\int \mathrm{d}x\,\mathrm{d}y\left(\int_{-b(N+1)}^{b(N+1)} c(N+1)\,\mathrm{d}\eta-\int_{-b(N)}^{b(N)} c(N)\,\mathrm{d}\eta\right), \tag{18}$$

where

$$c(N)=\{[g(N)]^2-\eta^2/[1-(a\lambda_c)^2N/(H/H_c)]^2\}^{1/2}.$$

With this expression for the density of states and explicitly writing out the summation over the quantum number n, we get for the thermodynamic potential density the following expression:

$$\begin{aligned}\sigma=-\frac{kT}{(2\pi\lambda_c)^3}\Bigg[\tfrac{1}{2}\int_{-b(1)}^{b(1)} c(1)\,\mathrm{d}\eta\int_{-\infty}^{+\infty}\log[1+\exp\beta(\mu-\varepsilon_0)]\,\mathrm{d}\xi \\ +\sum_{N=1}\left(\int_{-b(N+1)}^{b(N+1)} c(N+1)\,\mathrm{d}\eta-\int_{-b(N)}^{b(N)} c(N)\,\mathrm{d}\eta\right) \\ \times\int_{-\infty}^{+\infty}\log[1+\exp\beta(\mu-\varepsilon_N)]\,\mathrm{d}\xi\Bigg],\end{aligned} \tag{19}$$

where $\xi=p_z/mc$. In writing equation (19) the spin summation has been performed by noting the two-fold spin degeneracy of the eigenvalues, except for the state $N=0$. It can be easily seen that in the limit $a\to 0$, ω_N reduces to the usual degeneracy factor of the homogeneous case (Canuto *et al* 1969).

The constraint on the quantum number N in (6) can be rewritten so as to give an upper limit for η:

$$|\eta|\leqslant(H/H_c)[1-(a\lambda_c)^2N/(H/H_c)]^2/(a\lambda_c)^2. \tag{20}$$

This, when taken together with the limits of the η integration, leads to the condition

$$H/H_c\geqslant(a\lambda_c)^2N(2+\sqrt{2}). \tag{21}$$

2.4. *The degenerate electron gas*

We are at first interested in calculating the magnetic moment of only the degenerate electron gas. The electron gas becomes degenerate when

$$\varepsilon_N<\mu \qquad \text{and} \qquad T\to 0. \tag{22}$$

Under this condition the expression $\log[1+\exp\beta(\mu-\varepsilon_N)]$ appearing in equation (19) is replaced by $(\mu-\varepsilon_N)\beta$; also the integration over ξ becomes finite with limits $-d(N)$ and $d(N)$ where

$$d(N)=\{\mu^2-1-\eta^2[1-(1-(a\lambda_c)^2N/(H/H_c))^{-2}]-[2N(H/H_c)-(a\lambda_cN)^2]\}^{1/2}. \tag{23}$$

Incorporating this into the expression for σ, we get in the degeneracy limit

$$\sigma = -\frac{mc^2}{(2\pi\lambda_c)^3}\Bigg[\tfrac{1}{2}\int_{-b(1)}^{b(1)} c(1)\,d\eta \int_{-d(0)}^{d(0)} (\mu - \varepsilon_0)\,d\xi + \sum_{N=1}^{K}\left(\int_{-b(N+1)}^{b(N+1)} c(N+1)\,d\eta - \int_{-b(N)}^{b(N)} c(N)\,d\eta\right)\int_{-d(N)}^{d(N)} (\mu - \varepsilon_N)\,d\xi\Bigg]. \quad (24)$$

It is to be noted that the summation over N terminates after some K which is the maximum value of N for which the radicand in (23) is positive.

To avoid singularities that would arise at the limits of the η integrations, when the expression for σ is differentiated with respect to H to obtain the magnetic moment density, we transform the variable limits occurring in (24) into constant ones from -1 to $+1$ using the transformation (Davis and Rabinowitz 1975)

$$\theta_i = (U_{i-1} + L_{i-1})/2 + (U_{i-1} - L_{i-1})t_i/2. \quad (25)$$

Here $\theta_1 = \eta$ and $\theta_2 = \xi$, t_i are the new variables and U_{i-1} and L_{i-1} are the upper and lower limits of the θ_i integrations; the Jacobian of the transformation is

$$J = \prod_{i=1,2} (U_{i-1} - L_{i-1})/2. \quad (26)$$

Hence we have

$$\sigma = -\frac{mc^2}{2(\pi\lambda_c)^3}\Bigg(\tfrac{1}{2}\int_{-1}^{-1}\int_{-1}^{+1} dt_1\,dt_2(\mu - \varepsilon'_0)\alpha_1\beta_1\gamma'_0 + \sum_{N=1}^{K}\int_{-1}^{+1}\int_{-1}^{+1} dt_1\,dt_2[(\mu - \varepsilon'_N)\alpha_{N+1}\beta_{N+1}\gamma'_N - (\mu - \varepsilon''_N)\alpha_N\beta_N\gamma_N]\Bigg), \quad (27)$$

where

$$\alpha_N = b(N), \quad (27a)$$

$$\beta_N = \rho_N^{1/2}(1 - t_1^2)^{1/2}, \quad (27b)$$

$$\gamma_N = [\mu^2 - 1 - \rho_N(\tau_N^2 - 1)t_1^2 - \rho_N]^{1/2}, \quad (27c)$$

$$\gamma'_N = [\mu^2 - 1 - \rho_{N+1}\tau_{N+1}^2(1 - \tau_N^{-2})t_1^2 - \rho_N]^{1/2}, \quad (27d)$$

$$\varepsilon'_N = \{(1 - t_2^2)[\tau_{N+1}^2\rho_{N+1}(1 - \tau_N^{-2})t_1^2 + \rho_N + 1] + \mu^2 t_2^2\}^{1/2} \quad (27e)$$

$$\varepsilon''_N = \{(1 - t_2^2)[\rho_N((\tau_N^2 - 1)t_1^2 + 1) + 1] + \mu^2 t_2^2\}^{1/2}. \quad (27f)$$

In the above equations

$$\rho_N = 2N(H/H_c) - (a\lambda_c N)^2 \quad (27g)$$

and

$$\tau_N = 1 - (a\lambda_c)^2 N/(H/H_c). \quad (27h)$$

2.5. *Expression for the magnetic moment*

We can now obtain the magnetic moment density I_z as given by (12):

$$I_z(y) = -\partial\sigma/\partial H_z(y). \quad (28)$$

Since the energy ε_N and hence σ involve $H_z(y)$ only through the parameters H and

a, we should vary σ only with respect to these latter quantities. Notice that σ does not depend on y, unlike H_z which is a function of y also, leading to the magnetic moment density I_z being a function of y. We prefer to fix the value of a and vary σ with respect to H only. Then we have

$$I_z(y) = -\partial\sigma/\partial H_z(y) = -(\partial\sigma/\partial H)(\partial H/\partial H_z(y)). \tag{29}$$

From (1) we have $\partial H/\partial H(y) = \cosh^2(ay)$. Therefore

$$I_z(y) = -(\partial\sigma/\partial H)\cosh^2(ay). \tag{30}$$

The quantity $\partial\sigma/\partial H$ is obtained from equations (27), 27(a)–(h) by differentiation with respect to H:

$$\begin{aligned}\frac{\partial\sigma}{\partial H} = -\frac{mc^2}{2(\pi\lambda_c)^3}\bigg(\tfrac{1}{2}\int_{-1}^{+1}\int_{-1}^{+1} &\mathrm{d}t_1\,\mathrm{d}t_2(\mu-\varepsilon_0')\\ &\times[(\partial\alpha_1/\partial H)\beta_1\gamma_0' + \alpha_1(\partial\beta_1/\partial H)\gamma_0']\\ &+\sum_{N=1}^{K}\int_{-1}^{+1}\int_{-1}^{+1}\mathrm{d}t_1\,\mathrm{d}t_2\,[\![\{(\mu-\varepsilon_N')[(\partial\alpha_{N+1}/\partial H)\beta_{N+1}\gamma_N'\\ &+\alpha_{N+1}(\partial\beta_{N+1}/\partial H)\gamma_N' + \alpha_{N+1}\beta_{N+1}(\partial\gamma_N'/\partial H)] - (\partial\varepsilon_N'/\partial H)\alpha_{N+1}\beta_{N+1}\gamma_N'\}\\ &-\{(\mu-\varepsilon_N'')[(\partial\alpha_N/\partial H)\beta_N\gamma_N + \alpha_N(\partial\beta_N/\partial H)\gamma_N + \alpha_N\beta_N(\partial\gamma_N/\partial H)]\\ &-(\partial\varepsilon_N''/\partial H)\alpha_N\beta_N\gamma_N\}]\!]\bigg).\end{aligned} \tag{31}$$

Substituting (31) into (30), we get the desired expression for the magnetic moment density. The magnetic moment M of the gas of volume V is given by (see equation (13))

$$M = -\int_V (\partial\sigma/\partial H)\cosh^2(ay)\,\mathrm{d}V, \tag{32}$$

where $(\partial\sigma/\partial H)$ is given by (31). When the integration is performed over a spherical region of radius r, over which the gas extends, we get

$$M = -\left(\frac{\partial\sigma}{\partial H}\right)\pi r^3\left(\frac{2}{3} + \frac{\cosh(2ar)}{2a^2r^2} + \frac{\sinh(2ar)}{4a^3r^3}\right). \tag{33}$$

In the limit $a \to 0$ the quantities inside the second parentheses reduce to $\frac{4}{3}$ as should be expected, when the field is homogeneous. It may be remarked that the appearance of the function $\cosh^2(ay)$ in the expression for the magnetic moment density in equation (30) does not lead to an unbounded increase of the magnetic moment. Though $\cosh^2(ay)$ increases with y, it remains finite for systems of interest. The dimensions of the systems over which the electron gas extends set a limit on the maximum value of the magnetic moment density and the total magnetic moment.

3. Numerical results and conclusion

We have computed the magnetic moment density $I_z(y)$ for various values of the field strength and the chemical potential μ. The results show that except for the space dependent factor $\cosh^2(ay)$, the magnetic moment density is almost the same as the magnetisation in the homogeneous case. In table 1 we give the value of $I_z(y)$ for a set of values of μ when $ay = 5.0$ and $H/H_c = 0.1$. It is observed that when $H/H_c = 1$ and 0.5, for the range of values of μ : 1.1 to 2.7, there is no transition to diamagnetism. In these cases the maximum value of N at which the summation in (31) terminated due to the degeneracy condition (22) is just 7. But for $H/H_c = 0.1$, from the table it can be seen that as higher quantum levels get filled up, the transition is more prevalent.

Table 1. A set of values of the chemical potential (in units of mc^2) and the corresponding values of the magnetisation (homogeneous case) and magnetic moment density (inhomogeneous case at $ay = 5.0$) (both in units of μ_B/λ_c^3) are listed for $H/H_c = 0.1$. Here $N = K$ is the quantum number at which the summation terminates (cf equation (31)).

Chemical potential μ	Magnetic moment density I: Homogeneous case $\times 10^{-3}$	Magnetic moment density I: Inhomogeneous case	$N = K$
1.1	0.310	1.71	2
1.5	−0.072	−0.397	7
2.0	4.47	24.6	15
2.5	−0.812	−4.48	27
3.0	7.52	41.4	40
3.5	−1.62	−8.91	57
4.0	10.2	56.1	75
4.5	−1.93	−10.6	97
5.0	13.6	74.8	120
5.5	−2.12	−11.6	147
6.0	16.0	88.0	175
6.5	−3.09	−16.2	207
7.0	17.6	96.8	240
7.5	−5.15	−28.3	277

The magnetic induction $B(y)$ is related to the impressed magnetic field $H(y)$ and the magnetic moment density $I(y)$ by the relation

$$B(y) = \mu'[H(y) + I(y)], \tag{34}$$

where μ' is the permeability of the medium. In order that spontaneous magnetisation occurs, the induced magnetic field $B(y)$ must be greater than the impressed field, or equivalently $I(y) > H(y)$. For the particular magnetic field under consideration this is in fact the case. For example, when $a = 5 \times 10^{-6}\ \text{cm}^{-1}$, $H/H_c = 1$ and $\mu = 2.0$, the magnetic moment density at the point $y = 10^6$ cm is

$$I(y = 10^6\ \text{cm}) = 4.435 \times 10^2 (\mu_B/\lambda_c^3)\text{G} = 7.144 \times 10^{13}\ \text{G}.$$

At this point the magnetic field strength is

$$H(y = 10^6\ \text{cm}) = H_c\, \text{sech}^2(5.0)\text{G} = 8.011 \times 10^9\ \text{G}.$$

Thus the magnetic moment density is greater than the inducing magnetic field by a factor of 10^4. Following the argument of Canuto and Chiu (1968c), we may conclude that an electron gas in this inhomogeneous magnetic field exhibits spontaneous magnetisation.

To the best of our knowledge there does not exist any alternative method in the literature for the study of the magnetic behaviour of an electron gas in any IMFS. We believe that our approach in the present paper can be usefully applied to study similar problems in other inhomogeneous magnetic fields and also in combined electric and magnetic fields.

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